



Singular Values and Singular Vectors

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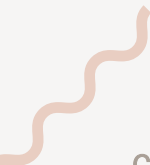
Introduction

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Singular Value

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Singular Vector



01

Introduction



range

null space

eigen value

eigen vector

transpose

inverse

symmetric matrix

orthogonal matrix

psd matrix



PCA

low-rank approximation

TLS minimization

pseudoinverse

separable models

optimal rotation

...

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

02

Singular Value



Singular Value Definition

- $S_{m \times n}$ Non-Square!!
- $\sigma_i = \sqrt{\lambda_i} \quad \lambda_i \in \sigma(S^T S), i = 1, \dots, n$
- $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{m-1} \geq \sigma_m$

Example

$$S = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

$$S^T S = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} \Rightarrow \lambda(S^T S) = \{360, 90, 0\}$$

$$\Rightarrow \begin{cases} \sigma_1 = \sqrt{360} = 6\sqrt{10} \\ \sigma_2 = \sqrt{90} = 3\sqrt{10} \\ \sigma_3 = 0 \end{cases}$$

Eigenvector of a Symmetric Matrix

Theorem

Orthogonality of Eigenvectors of a Symmetric Matrix Corresponding to Distinct Eigenvalues. If A is symmetric, then any two eigenvectors from different eigenspace are **orthogonal**.

$$\left. \begin{array}{l} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \\ \lambda_1 \neq \lambda_2 \end{array} \right\} \Rightarrow v_1^T v_2 = 0$$

Proof:

Singular Value Correlation to Norm

Theorem

$\{v_1, \dots, v_n\}$ are orthonormal eigenvectors of matrix $S^T S$ then singular values of matrix S are norm of Sv_i vectors:

$$\|Sv_i\| = \sigma_i$$

Proof?

Singular Value

Example

$$S = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \rightarrow S^T S = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix} \rightarrow \sigma_1 = \sqrt{360}, \sigma_2 = \sqrt{90}, \sigma_3 = 0$$

$$v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, v_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$Sv_1 = \begin{bmatrix} 18 \\ 6 \end{bmatrix} \Rightarrow \|Sv_1\| = \sqrt{18^2 + 6^2} = \sigma_1$$

$$Sv_2 = \begin{bmatrix} 3 \\ -9 \end{bmatrix} \Rightarrow \|Sv_2\| = \sqrt{3^2 + (-9)^2} = \sigma_2$$

$$Sv_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \|Sv_3\| = 0 = \sigma_3$$

03

Singular Vector



Singular Vector

Theorem

$\{v_1, \dots, v_n\}$ are orthonormal eigenvectors of matrix $S^T S$ and S has r non-zero singular value:

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0, \quad \sigma_{r+1} = \dots = \sigma_n = 0$$

$\{Sv_1, \dots, Sv_r\}$ is an orthogonal basis for range of S

$\text{rank}(S)=r$

Rank of Matrix = Number of nonzero singular values

How to find $\{u_1, \dots, u_r\}$ is an orthonormal basis for range of S

Singular Value & Singular Vector

How to find $\{u_1, \dots, u_r\}$ as an orthonormal basis for range of S ?

$$Sv = \sigma u$$

Diagram illustrating the equation $Sv = \sigma u$ with labels:

- S is labeled "Singular Value" (via an arrow from the label below).
- v is labeled "Singular Vector" (via an arrow from the label below).
- σ is labeled "Singular Value" (via an arrow from the label below).
- u is labeled "Singular Vector" (via an arrow from the label below).